Matrices

A **matrix** is a rectangular array of numbers. For example, the following rectangular arrays of numbers are matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 8 & 10 \end{bmatrix} \qquad C = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \\ 10 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 3 & 5 & 7 & 9 \end{bmatrix} \qquad E = \begin{bmatrix} 4 \\ 1 \\ 47653 \end{bmatrix}$$

Matrices vary in **size**. An $m \times n$ matrix has m rows and n columns. The matrices above have sizes

$$2 \times 2, \quad 2 \times 3, \quad 5 \times 1, \quad 1 \times 5, \quad 3 \times 1$$

respectively.

The numbers in the matrix are called the **entries** of the matrix. Because we may have the same number in more than one position, when we refer to an entry we refer to its position. The (i, j) entry is the entry in the ith row and jth column or the symbol A_{ij} denotes the entry in the i th row and j th column of the matrix A.

Example Using the matrices shown above:

$$A_{12} = 2, \quad A_{21} = 3, \quad C_{31} = 6, \quad B_{23} = 10, \quad E_{31} = 47653.$$

Example Using the matrices defined above find:

$$A_{22} =$$

$$B_{12} =$$

$$D_{13} =$$

$$E_{21} =$$

Matrices arise naturally in many areas of mathematics. They are especially useful in situations where we have cross classification since the array format allows us to list all possibilities compactly. In fact we have already used arrays or tables such as these in the calculation of conditional probabilities. This will also be especially useful in game theory.

Algebra of Matrices

Matrices have arithmetic properties, just like ordinary numbers. We can define an addition and a multiplication for matrices. In both of these binary operations there will be compatibility restrictions on the sizes of the matrices involved.

Adding Two matrices

Before we can add two matrices, they must have the same size. Any two matrices of the same size can be added. We add matrices by adding the corresponding entries. For example:

$$\begin{bmatrix} 2 & 1 & 0 \\ 4 & 0 & 1 \\ 1 & 2 & 3 \\ 0 & 3 & 10 \end{bmatrix} + \begin{bmatrix} 0 & 5 & 7 \\ 4 & 3 & 1 \\ 2 & 2 & 1 \\ 9 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 2+0 & 1+5 & 0+7 \\ 4+4 & 0+3 & 1+1 \\ 1+2 & 2+2 & 3+1 \\ 0+9 & 3+4 & 10+0 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 7 \\ 8 & 3 & 2 \\ 3 & 4 & 4 \\ 9 & 7 & 10 \end{bmatrix}$$

So, if we add two matrices, A and B, the (i, j) entry of $A + B$ is equal to $A_{ij} + B_{ij}$.

Example Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$ $\quad D = \begin{bmatrix} 1 & 0 \\ 5 & 9 \end{bmatrix}$

Then $\quad A + D =$

Matrix Multiplication (A mild form)

We start by multiplying a row matrix by a column matrix. Here the number of entries in the row must equal the number of entries in the column. The general formula is given by

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} = a_1 \cdot b_1 + a_2 \cdot b_2 \ \cdots \ a_{n-1} \cdot b_{n-1} + a_n \cdot b_n$$

Example Calculate the following:

$$\begin{bmatrix} 1 & 2 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 1 \\ 5 \\ 3 \end{bmatrix} = \qquad \begin{bmatrix} 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} =$$

$$\begin{bmatrix} 3 & 8 & 7 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 5 \end{bmatrix} = \qquad \begin{bmatrix} 1 & 8 & 4 & 5 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \\ 1 \\ 1 \end{bmatrix} =$$

Note that when you multiply a row by a column, you just get a number or a 1×1 matrix. Above we have (in order),

A 1×5 matrix multiplied by a 5×1 matrix gives a 1×1 matrix.
A 1×3 matrix multiplied by a 3×1 matrix gives a 1×1 matrix.
A 1×4 matrix multiplied by a 4×1 matrix gives a 1×1 matrix.
A 1×6 matrix multiplied by a 6×1 matrix gives a 1×1 matrix.

Compatibility In order to multiply two matrices, we must have compatible sizes. Let A be an $m \times p$ matrix and let B be a $q \times n$ matrix. Then I can form the product AB only if $p = q$. If $p = q$, then AB will be an $m \times n$ matrix.

Example: let:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 8 & 10 \end{bmatrix} \qquad C = \begin{bmatrix} 2 \\ 4 \\ 6 \\ 8 \\ 10 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 3 & 5 & 7 & 9 \end{bmatrix} \qquad E = \begin{bmatrix} 4 \\ 1 \\ 47653 \end{bmatrix}$$

Which of the following matrix products can be formed and if it can be formed, what size is the matrix?

Product	AB	BC	AC	DC	AE	EA	EB	BE
Possible Y/N								
Size								

Rather than study general matrix multiplication, we will limit our study of matrix multiplication to that which will occur in game theory. Our goal is to be able to calculate products of the form:

$$AB \qquad BC \qquad \text{and} \qquad ABC,$$

where A is a $1 \times m$ row matrix, B is an $m \times n$ matrix and C is a column matrix of the form $n \times 1$. Because of associativity of matrix multiplication, the latter product ABC can be calculated in either of two ways, as $(AB)C$ or as $A(BC)$.

To multiply the $1 \times m$ row matrix A by the $m \times n$ matrix B, we multiply the row matrix A by the columns of B to get the entries of AB. Specifically, AB is a $1 \times n$ matrix (a row matrix) the $(1, j)$ entry of AB is the row matrix A multiplied by the j th column of B.

Example Let $A = \begin{bmatrix} 1 & 2 \end{bmatrix}$ $\qquad B = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 8 & 10 \end{bmatrix}$

Since A is a 1×2 matrix and B is a 2×3 matrix, AB will be a 1×3 matrix.

To calculate the $(1, 1)$ entry of AB, we multiply the row matrix A by column 1 of B

$$\overset{A}{\begin{bmatrix} 1 & 2 \end{bmatrix}} \overset{B}{\begin{bmatrix} 2 & 4 & 7 \\ 5 & 8 & 10 \end{bmatrix}} = \overset{AB}{\begin{bmatrix} 1 \cdot 2 + 2 \cdot 5 & - & - \end{bmatrix} = \begin{bmatrix} 12 & - & - \end{bmatrix}}$$

To calculate the $(1, 2)$ entry of AB, we multiply the row matrix A by Column 2 of B.

$$\overset{A}{\begin{bmatrix} 1 & 2 \end{bmatrix}} \overset{B}{\begin{bmatrix} 2 & 4 & 7 \\ 5 & 8 & 10 \end{bmatrix}} = \overset{AB}{\begin{bmatrix} 12 & 1 \cdot 4 + 2 \cdot 8 & - \end{bmatrix} = \begin{bmatrix} 12 & 20 & - \end{bmatrix}}$$

To calculate the $(1, 3)$ entry of AB, we multiply Row 1 of A by Column 3 of B.

$$\overset{A}{\begin{bmatrix} 1 & 2 \end{bmatrix}} \overset{B}{\begin{bmatrix} 2 & 4 & 7 \\ 5 & 8 & 10 \end{bmatrix}} = \overset{AB}{\begin{bmatrix} 12 & 20 & 1 \cdot 7 + 2 \cdot 10 \end{bmatrix} = \begin{bmatrix} 12 & 20 & 27 \end{bmatrix}}$$

Example Let

$$A = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ 4 & 1 \end{bmatrix}$$

Calculate AB

To multiply the $m \times n$ matrix B by the $n \times 1$ column matrix C, we multiply each row of B by the column matrix C to get the rows of BC. In particular, BC is a $m \times 1$ column matrix where the $(k, 1)$ entry of BC is the kth row of A multiplied by the column matrix B.

Example Let $B = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 8 & 10 \end{bmatrix}$ and $C = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

To calculate the $(1, 1)$ entry of BC, we multiply row 1 of B by the column matrix C.

$$\overset{B}{\begin{bmatrix} 2 & 4 & 7 \\ 5 & 8 & 10 \end{bmatrix}} \overset{C}{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}} = \begin{bmatrix} 2 \cdot 1 + 4 \cdot 0 + 7 \cdot (-1) \\ - \end{bmatrix} = \overset{BC}{\begin{bmatrix} -5 \\ - \end{bmatrix}}$$

To calculate the $(2, 1)$ entry of BC, we multiply row 2 of B by the column matrix C.

$$\overset{B}{\begin{bmatrix} 2 & 4 & 7 \\ 5 & 8 & 10 \end{bmatrix}} \overset{C}{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}} = \begin{bmatrix} -5 \\ 5 \cdot 1 + 8 \cdot 0 + 10 \cdot (-1) \end{bmatrix} = \overset{BC}{\begin{bmatrix} -5 \\ -5 \end{bmatrix}}$$

Example Let $B = \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ 4 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Calculate BC.

Example Let Let $A = \begin{bmatrix} 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 8 & 10 \end{bmatrix}$ and $C = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

To Calculate ABC, we can calculate $A(BC)$ or $(AB)C$.

By our calculations above:

$$A(BC) = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} -5 \\ -5 \end{bmatrix} = 1 \cdot (-5) + 2 \cdot (-5) = -15.$$

$$(AB)C = \begin{bmatrix} 12 & 20 & 27 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 12 \cdot 1 + 20 \cdot 0 + 27 \cdot (-1) = -15.$$

(Obviously we need only do one of the above calculations in order to calculate ABC = -15.)

Example Let

$$A = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ 4 & 1 \end{bmatrix} \text{ and } \quad C = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Calculate ABC.

An application to previous material

This section will not appear on any exam in this course but it is interesting and a useful way to remember certain formulas. It also minimizes the actual amount calculation you need to do in certain problems.

Given two column vectors ($r \times 1$ sized matrices), A and B, define a number $A \bullet B$ by turning A on its side and multiplying the two matrices. For example if $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

$$[A \bullet B] = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = [1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6] = [4 + 10 + 18] = [32]$$

so $A \bullet B = 32$. If you do this a lot you see that there is no need to turn matrices on their sides: just right one next to the other, multiply the first rows, add it to the product of the second rows, add this to the product of the third rows and so on. Excel calls the dot product the SUMPRODUCT. In the example

$$A \bullet B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \bullet \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 4 + 10 + 18 = 32$$

Now for the application to expected value and variance of a random variable from a relative frequency or probability distribution table.

Example The rules of a carnival game are as follows:

1. The player pays \$1 to play the game.
2. The player then flips a fair coin, if the player gets a head the game attendant gives the player \$2 and the player stops playing.
3. If the player gets a tail on the coin, the player rolls a fair six-sided die. If the player gets a six, the game attendant gives the player \$1 and the game is over.
4. If the player does not get a six on the die, the game is over and the game attendant gives nothing to the player.

Let X denote the player's (net)earnings for this game, last day, we saw that the probability distribution of X is given by:

X	**P(X)**
-1	5/12
0	1/12
1	1/2

Use the value for $\mu = E(X)$ found above to find the variance and standard deviation of X, that is find $\sigma^2(X)$ and $\sigma(X)$.

$\mathbf{x}_i$	$\mathbf{p}_i$	$\mathbf{x}_i \cdot \mathbf{p}_i$	$(\mathbf{x}_i - \mu)$	$(\mathbf{x}_i - \mu)^2$	$\mathbf{p}_i \cdot (\mathbf{x}_i - \mu)^2$
-1	5/12	$\frac{-5}{12}$	$\frac{-13}{12}$	$\frac{169}{144}$	$\frac{845}{1728}$
0	1/12	$\frac{0}{12}$	$\frac{-1}{12}$	$\frac{1}{144}$	$\frac{1}{1728}$
1	6/12	$\frac{6}{12}$	$\frac{11}{12}$	$\frac{121}{144}$	$\frac{726}{1728}$
		Sum $= \mu = \frac{1}{12}$			**Sum** $= \sigma^2(X) = \frac{1572}{1728} \approx 0.9097222222$

$\sigma \approx 0.9537935952.$

Same example with matrices. Let $X = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $P = \begin{bmatrix} \frac{5}{12} \\ \frac{1}{12} \\ \frac{6}{12} \end{bmatrix}$

Compute $X \bullet P = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} \frac{5}{12} \\ \frac{1}{12} \\ \frac{6}{12} \end{bmatrix} = -1 \cdot \frac{5}{12} + 0 \cdot \frac{1}{12} + 1 \cdot \frac{6}{12} = \frac{1}{12}$ so

$$E(X) = \frac{1}{12}$$

So far this is just an easy way to remember the formula for the expected value: just take the dot product of the outcome column (the $\mathbf{x}_i$'s) and the probability distribution column (the $\mathbf{p}_i$'s).

To compute the variance, let us introduce a formula which makes its calculation much less work. It follows from algebra manipulations (basically $(x+y)^2 = x^2 + 2xy + y^2$) that

$$\sigma^2 = E(X^2) - \big(E(X)\big)^2$$

To compute the variance for the example we are working on, replace X with the column vector of squares $Y = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Then $E(X^2) = Y \bullet P = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} \frac{5}{12} \\ \frac{1}{12} \\ \frac{6}{12} \end{bmatrix} = 1 \cdot \frac{5}{12} + 0 \cdot \frac{1}{12} + 1 \cdot \frac{6}{12} = \frac{11}{12}$ so

$$\sigma^2 = E(X^2) - \big(E(X)\big)^2 = \frac{11}{12} - \left(\frac{1}{12}\right)^2 = \frac{11 \cdot 12 - 1}{12^2} = \frac{131}{144} = \frac{1572}{1728}$$